Model Reduction and Control of Flexible Structures Using Krylov Vectors

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Krylov vectors and the concept of parameter matching are combined together to develop a model reduction algorithm for a damped structural dynamics system. The obtained reduced-order model matches a certain number of low-frequency moments of the full-order system. The major application of the present method is to the control of flexible structures. It is shown that, in the control of flexible structures, three types of control energy spillover generally exist: control, observation, and dynamic. The formulation based on Krylov vectors can eliminate both the control and observation spillovers while leaving only the dynamic spillover to be considered. Two examples are used to illustrate the efficacy of the Krylov method.

I. Introduction

A MAJOR difficulty in the control of flexible structures or any other large-scale system is, in the words of Bellman, the "curse of dimensionality." A flexible structure is, by nature, a distributed-parameter system, and, hence, it has infinitely many degrees of freedom. Even approximate structural models obtained by some discretization approach are generally still too large for use in control design applications. Therefore, model order reduction plays an indispensable role in the control of flexible structures. Usually, model reduction of a structural dynamics system is performed by the Rayleigh-Ritz method, which transforms the system equation to a smaller scale by using a projection subspace. It is indisputable that the choice of projection subspace is important to the accuracy of the reduced model. The eigensubspace, or the normal mode subspace, is frequently used for projection because it has a clear physical meaning and can preserve the system natural frequencies. However, with regard to the accuracy of system response, numerical experience has shown that preservation of the natural frequencies is usually not the only concern. Other than normal modes, there are other static modes, e.g., constraint modes, attachment modes, and inertia-relief modes, which are frequently used in component mode synthesis.[1] In this paper, Krylov vectors, which can be considered as static modes, are used for model reduction. There has been quite a bit of research concerning the convergence and efficiency of Krylov vectors in application to eigenvalue analysis and to the structural dynamics model reduction problem.[2-5] Krylov vectors are also efficient when employed in general linear system and controller reduction problems.[6,7] The major purpose of this paper is to discuss the possible application of Krylov vectors to controller design for flexible structures.

The structural dynamics system studied here is described by a second-order matrix differential equation together with an output measurement equation. To perform model reduction to a structural dynamics system in this input-output configuration, the concept of parameter matching for general linear system model reduction is adopted. Parameter matching constitutes a class of efficient methods for model order reduction of general linear systems.[8-12] A parameter-matching method constructs a reduced-order model that matches a certain number of system parameters, for instance, low-frequency moments and/or high-frequency moments. However, all previous parameter-matching methods deal with systems described either by transfer functions or by the first-order state-space form. To apply the existing parameter-matching methods to a structural dynamics system, it is necessary to put the system equation into first-order form. There is some disadvantage to this approach, namely, the symmetry and physical meaning of the system matrices is destroyed. Therefore, one contribution of this paper is that, for the first time, Krylov vectors and the concept of parameter matching are combined together to develop a model reduction algorithm for a second-order system. The reduced-order model obtained has the property of parameter matching. Thus, Krylov vectors not only have significant physical meaning but also have this association with well-known model reduction methods. Although there are two sets of system parameters, low-frequency moments and high-frequency moments, the present method matches low-frequency moments. In this case, the reduced-order model aproximates the lower natural frequencies of the full-order model.

The second contribution of this paper is that, to the authors' knowledge, this is the first time that damping has been directly taken into consideration in generating the Krylov vectors of a second-order system model. All previous model reduction methods based on Krylov vectors include only the mass and stiffness matrices in the algorithm. Reference 6 shows that the frequently used Krylov vectors can indeed produce a reduced-order model with the moment-matching property for an undamped structural system. However, if the Krylov vectors generated for the undamped system are applied to the damped system, the reduced system no longer possesses the parameter-matching property. Therefore, a more appropriate algorithm should include the damping effect. A Krylov algorithm for damped structural systems is proposed in this paper.

Finally, the major application of the present model reduction method is to the control of flexible structures. A basic topic in the control of flexible structures is how to reduce the spillover of control energy, which is a direct result of model reduction. Spillover of control energy from the controlled subsystem into the residual subsystem usually degrades the controller performance and sometimes may lead to stability problems. It is noted in this paper that there are three types of spillover: control, observation, and dynamic. The transformed system equation in Krylov coordinates turns out to have dynamic spillover but no control or observation spillover, which is the major difference between this method and the normal mode method.

Received Jan. 23, 1989; presented as Paper 89-1237-CP at the AIAA/ASME/ASCE/AHS/ASC 30th Structures, Structural Dynamics, and Materials Conference, Mobile, AL, April 3-5, 1989; revision received Oct. 15, 1989.

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This paper is organized as follows. In Sec. II, first the parameter-matching method for general linear systems is briefly reviewed; then the parameter-matching model reduction algorithm for structural dynamics systems in second-order form is developed. In Sec. III, the characteristics of spillover are depicted, and a Krylov formulation for control of flexible structures is derived. In Sec. IV, a model reduction example and a flexible structure control example are used to illustrate the efficacy of the Krylov method. This paper is concluded in Sec. V.

II. Model Order Reduction by Matching System Parameters

In this section, we first review briefly the parameter-matching model order reduction method for general linear systems described in first-order state-space form. Then we extend the concept of parameter matching to a structural dynamics system, which is described by a second-order matrix differential equation and an output measurement equation.

A. General Linear Systems

An nth order, linear, time-invariant system is described by

$$\dot{z} = Az + Bu, \qquad z \in R^n, \qquad u \in R^l$$

$$y = Cz, \qquad y \in R^m \tag{1}$$

for which the transfer function $G(s) = C(sI - A)^{-1}B$ can be formally expanded in a Laurent series around $s = \infty$ as

$$G(s) = \sum_{i=0}^{\infty} CA^iBs^{-i-1} \tag{2}$$

If the system has no pole at the origin, then the Taylor series expansion of $G(s)$ around $s = 0$ yields

$$G(s) = \sum_{i=0}^{\infty} -CA^{-i-1}Bs^i \tag{3}$$

From Eq. (2), we get a set of system parameters $\{CA^iB \mid i = 0,1,\ldots\}$, which are termed Markov parameters[9,11] or high-frequency moments.[12] From Eq. (3), we get another set of system parameters $\{CA^{-i}B \mid i = 1,2,\ldots\}$, which are termed time moments[9,11] or low-frequency moments.[12] These two sets of parameters constitute pieces of system data for the triple (A, B, C). They provide a data base for system identification [for instance, the Eigensystem Realization Algorithm (ERA) method[13]]. In the model reduction area, one school of approach, called the parameter-matching method, seeks to construct a reduced-order model such that it matches a certain number of parameters of the full-order system. In Ref. 12 Villemagne and Skelton provide a toolbox for producing parameter-matching reduced-order models. The reduced-order model is obtained by an oblique projection approach and is in the form

$$\dot{z}_R = A_R z_R + B_R u \qquad z_R \in R^r$$

$$y = C_R z_R \tag{4}$$

where $r < n$, $A_R = TAR$, $B_R = TB$, $C_R = CR$, and $TR = I_r$, with T and R the left and right projection matrices. It is shown in Ref. 12 that if T and R are chosen such that span$[T]$ = span$[(A^T)^{-p}C^T, (A^T)^{-p+1}C^T, \ldots, (A^T)^qC^T]$ and span$[R]$ = span$[A^{-s}B, A^{-s+1}B, \ldots, A^tB]$ with $p, q, s, t \geq 0$ and $p + q = s + t$, then the reduced-order model matches $p + s$ low-frequency moments and $q + t$ high-frequency moments. That is, $C_R A_R^i B_R = CA^iB$, for $i = -p-s, \ldots, q+t$.

B. Structural Dynamics Systems

A structural dynamics system can be described by the input-output form

$$M\ddot{x} + D\dot{x} + Kx = Pu \tag{5a}$$

$$y = Vx + W\dot{x} \tag{5b}$$

where $x \in R^n$ is the displacement vector; $u \in R^l$ the input vector; $y \in R^m$ the output measurement vector; M, D, and K the system mass, damping, and stiffness matrices, respectively; and V and W the displacement and velocity sensor distribution matrices, respectively. The damping matrix can be general damping, but it is assumed to be symmetric. In most practical cases, we can assume that l and m are much smaller than n.

The frequency response solution of Eq. (5a) is $X(\omega) = (K + j\omega D - \omega^2 M)^{-1}PU(\omega)$, with $X(\omega)$ and $U(\omega)$ the Fourier transforms of x and u. If the system is assumed to have no rigid-body modes, then the output frequency response can be formally represented by a Taylor series

$$\begin{aligned} Y(\omega) &= (V + j\omega W)(-\omega^2 M + j\omega D + K)^{-1}PU(\omega) \\ &= (V + j\omega W)(I + j\omega K^{-1}D - \omega^2 K^{-1}M)^{-1}K^{-1}PU(\omega) \\ &= \sum_{i=0}^{\infty} (V + j\omega W)(\omega^2 K^{-1}M - j\omega K^{-1}D)^i K^{-1}PU(\omega) \\ &= \Big\{VK^{-1}P + j\omega(W - VK^{-1}D)K^{-1}P + \omega^2\big[VK^{-1}M \\ &\quad + WK^{-1}D - V(K^{-1}D)^2\big]K^{-1}P + \cdots\Big\}U(\omega) \end{aligned} \tag{6}$$

The low-frequency moments are defined as the coefficient matrices in the previous expansion series.

To arrive at an algorithm for constructing a reduced-order model that matches low-frequency moments, it is easier to start from the first-order formulation. The first-order differential equation equivalent to Eqs. (5) can be expressed as

$$\begin{bmatrix} D & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix} u$$

$$y = [V \quad W] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \tag{7}$$

or

$$\hat{M}\dot{z} + \hat{K}z = \hat{P}u$$

$$y = \hat{V}z \tag{8}$$

with

$$\hat{M} = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \qquad \hat{K} = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} P \\ 0 \end{bmatrix}, \qquad \hat{V} = [V \quad W] \tag{9}$$

In standard state-space form, the system equation is

$$\dot{z} = -\hat{M}^{-1}\hat{K}z + \hat{M}^{-1}\hat{P}u$$

$$y = \hat{V}z \tag{10}$$

Recalling that $CA^{-i}B$ is the low-frequency moment for a system described by Eq. (1), we can notate the low-frequency moments for the preceding system as

$$T_i = \hat{V}(-\hat{M}^{-1}\hat{K})^{-i}\hat{M}^{-1}\hat{P} = (-1)^i\hat{V}(\hat{K}^{-1}\hat{M})^{i-1}\hat{K}^{-1}\hat{P} \qquad i = 1, 2, \ldots \tag{11}$$

It can be shown that $\hat{V}(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$ is equal to the coefficient matrix associated with the $(j\omega)^{i-1}$ term in Eq. (6). This is not a coincidence, because the definition of low-frequency moments is based on the Taylor series expansion of the output frequency response, which is unique whether the formulation is of first order or second order.

To apply the existing parameter-matching model reduction methods to a structural dynamics system described by Eqs. (5), it would be necessary to start from the first-order state-space equation, Eq. (10). If this were done, then, of course, the reduced-system equation would be in first-order form. However, there are at least two reasons that this is not the best approach. First of all, there is no guarantee that the reduced-order model would be a stable system. This is, in fact, the major shortcoming of the moment-matching approach based on a first-order formulation. The second disadvantage of the first-order formulation is that the symmetry and the physical meaning of the system matrices would be destroyed. Structural dynamicists usually prefer a second-order formulation. That is why we seek a projection subspace that can be employed to perform model reduction on the second-order equation and can produce a reduced-order model with the moment-matching property.

Before formulating an algorithm for generating the projection subspace, we can use Eq. (11) to develop a Krylov recurrence procedure. Substitution of Eq. (9) into Eq. (11) gives

$$T_i = (-1)^i [V \quad W] \begin{bmatrix} -K^{-1}D & -K^{-1}M \\ I & 0 \end{bmatrix}^{i-1} \begin{bmatrix} K^{-1}P \\ 0 \end{bmatrix}$$

This suggests the following recurrence formula

$$\begin{bmatrix} Q_{j+1}^d \\ Q_{j+1}^v \end{bmatrix} = \begin{bmatrix} -K^{-1}D & -K^{-1}M \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_j^d \\ Q_j^v \end{bmatrix} \tag{12}$$

or, equivalently,

$$Q_{j+1}^d = -K^{-1}DQ_j^d - K^{-1}MQ_j^v = -K^{-1}DQ_j^d - K^{-1}MQ_{j-1}^d$$

$$Q_{j+1}^v = Q_j^d \tag{13}$$

Superscripts d and v denote displacement and velocity portions of the vector, respectively. The matrix containing the generated vector sequence is called a Krylov matrix and has the form

$$\begin{bmatrix} Q_1^d & Q_2^d & Q_3^d & \cdots \\ Q_1^v & Q_1^d & Q_2^d & \cdots \end{bmatrix}$$

According to the preceding form, we have the following theorem concerning the choice of projection subspace and the moment-matching property:

Theorem: Let

$$L_{\hat{P}} = \begin{bmatrix} Q_1^d & Q_2^d & Q_3^d & \cdots & Q_p^d \\ 0 & Q_1^d & Q_2^d & \cdots & Q_{p-1}^d \end{bmatrix}$$

be the Krylov matrix generated by Eq. (13) with $\hat{K}^{-1}\hat{P}$ the starting block of vectors, i.e., $Q_1^d = K^{-1}P$, $Q_1^v = 0$, and let

$$L_{\hat{V}} = \begin{bmatrix} P_1^d & P_2^d & P_3^d & \cdots & P_q^d \\ P_1^v & P_1^d & P_2^d & \cdots & P_{q-1}^d \end{bmatrix}$$

be the Krylov matrix generated by Eq. (13) with $\hat{K}^{-1}\hat{V}$ the starting block of vectors, i.e., $P_1^d = K^{-1}V^T$, $P_1^v = -M^{-1}W^T$. If $L = \text{span}[Q_1^d \cdots Q_p^d P_1^d \cdots P_q^d P_1^v]$ is chosen as the projection subspace for model reduction of the structural dynamics system described by Eqs. (5), then the reduced-order model matches $\hat{V}(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$, for $i = 0, 1, \ldots, p+q+1$.

Proof: The proof is recursive. First, the reduced-system equation is expressed in the form

$$L^TML\ddot{\bar{x}} + L^TDL\dot{\bar{x}} + L^TKL\bar{x} = L^TPu$$

$$y = VL\bar{x} + WL\dot{\bar{x}} \tag{14}$$

with $x = L\bar{x}$. If we define

$$\hat{L} = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$$

then the preceding equation can be rewritten into the form of Eq. (8)

$$\hat{L}^T\hat{M}\hat{L}\dot{\bar{z}} + \hat{L}^T\hat{K}\hat{L}\bar{z} = \hat{L}^T\hat{P}u$$

$$y = \hat{V}\hat{L}\bar{z} \tag{15}$$

for which the low-frequency moments, according to Eqs. (11) and (8), are

$$(\hat{V}\hat{L})[(\hat{L}^T\hat{K}\hat{L})^{-1}(\hat{L}^T\hat{M}\hat{L})]^{i-1}(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^T\hat{P}, \quad i = 1, 2, \cdots$$

Next, we have the following property: If a vector v is such that $\hat{K}^{-1}v$ is contained in $\hat{L}$, i.e., $\hat{K}^{-1}v = \hat{L}\alpha$, then

$$\hat{L}(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^Tv = \hat{L}(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^T\hat{K}(\hat{K}^{-1}v)$$

$$= \hat{L}(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^T\hat{K}\hat{L}\alpha = \hat{L}\alpha = \hat{K}^{-1}v$$

By using this property and the fact that $(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$, $i = 0, 1, \ldots, p$, and $(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{V}^T$, $i = 0, 1, \ldots, q$, are contained in $\hat{L}$, it can be shown that

$$\hat{L}[(\hat{L}^T\hat{K}\hat{L})^{-1}(\hat{L}^T\hat{M}\hat{L})]^i(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^TP = (\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$$

$$i = 0, 1, \cdots, p$$

$$\hat{V}\hat{L}[(\hat{L}^T\hat{K}\hat{L})^{-1}(\hat{L}^T\hat{M}\hat{L})]^i(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^T = \hat{V}(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}$$

$$i = 0, 1, \cdots, q$$

Therefore,

$$\hat{V}\hat{L}[(\hat{L}^T\hat{K}\hat{L})^{-1}(\hat{L}^T\hat{M}\hat{L})]^i(\hat{L}^T\hat{K}\hat{L})^{-1}\hat{L}^T\hat{P} = \hat{V}(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$$

for $i = 0, 1, 2, \cdots, p+q+1$. ■

The $L_{\hat{P}}$ and $L_{\hat{V}}$ matrices in the theorem are the generalized controllabilty and generalized observability matrices of the system [Eq. (10)]. For the purpose of response simulation only, we can choose L to be the d portion of $L_{\hat{P}}$. For control applications, as will be shown in Sec. III, it is desirable to construct a reduced-order model without control or observation spillover. Therefore, the following algorithm is proposed:

Algorithm:

1) Starting vector:

$$Q_0^d = Q_0^v = 0$$

$$R_0^d = K^{-1}\tilde{P}$$

$\tilde{P}$ = linearly independent portion of

$$[P \; V^T \; W^T(M^{-1}W^T)]$$

$$R_0^v = -M^{-1}W^T$$

$(R_0^d)^TKR_0 = U_0\Sigma_0U_0^T$ (singular-value decomposition)

$Q_1^d = R_0^dU_0\Sigma_0^{-1/2}$ (Q_1^d normalized w.r.t. K)

$$Q_1^v = R_0^vU_0\Sigma_0^{-1/2}$$

2) For $j = 1, 2, \ldots, k-1$, repeat:

$$\left.\begin{aligned} R_j^d &= -K^{-1}DQ_j^d - K^{-1}MQ_j^v \\ R_j^v &= Q_j^d \end{aligned}\right\} \text{(new vector)}$$

For $i = 1, 2, \ldots, j$, repeat:

$$\left.\begin{aligned} \alpha_i &= (Q_i^d)^T K R_j^d \\ R_j^d &= R_j^d - Q_i^d \alpha_i \\ R_j^v &= R_j^v - Q_i^v \alpha_i \end{aligned}\right\} \text{(orthogonalization)}$$

end;

$$(R_j^d)^T K R_j^d = U_j \Sigma_j U_j^T \quad \text{(singular-value decomposition)}$$

$$Q_{j+1}^d = R_j^d U_j \Sigma_j^{-1/2}$$
$$Q_{j+1}^v = R_j^v U_j \Sigma_j^{-1/2}$$

end;

3) Form the k-block projection matrix

$$L = \left[Q_1^v \quad Q_1^d \quad Q_2^d \quad \cdots \quad Q_{k-1}^d\right]$$

In the preceding algorithm, the starting vectors are chosen to produce a K-normalized L matrix and a transformed system equation with force and sensor distribution matrices having nonzero elements only in the first block. This special form of the transformed system equation has one major advantage in control applications, which will be discussed in Sec. III. The size of the block can be reduced if the actuator/sensor allocation is designed to be collocated, since $\hat{P}$ is the linearly independent portion of $[P \;\; V^T \;\; W^T \;\; (M^{-1} W^T)]$.

The reduced-order model is guaranteed to be stable if the damping matrix is positive definite. This can be proved by the Lyapunov theorem,[14] which says that if a positive-definite function $\mathcal{U}$, which is a function of the system states, can be found such that $\dot{\mathcal{U}}$ is negative-definite, then the system is asymptotically stable if the system is controllable. If we define the Lyapunov function $\mathcal{U}$ of the reduced system as the system's total energy

$$\mathcal{U} = (1/2)\left[\dot{\bar{x}}^T (L^T M L)\dot{\bar{x}} + \bar{x}^T (L^T K L)\bar{x}\right]$$

then,

$$\dot{\mathcal{U}} = \dot{\bar{x}}^T (L^T M L \ddot{\bar{x}} + L^T K L \bar{x}) = -\dot{\bar{x}}^T L^T D L \dot{\bar{x}} \leq 0$$

if D is positive-definite. The $\dot{\mathcal{U}} = 0$ holds only if $L\dot{\bar{x}} = 0$ or $\dot{\bar{x}} = 0$, since L contains linearly independent vectors. This may occur at some discrete instants of time but not over any finite interval of time, no matter how small. Therefore, the system energy decreases with time and eventually approaches zero. The reduced system is asymptotically stable.

III. Control of Flexible Structures

A major difficulty in the control of flexible structures comes from the spillover of control energy, which is a direct result of model order reduction.[15] Because of the distributed-parameter nature of flexible-structure systems, a distributed control theory together with distributed actuator and sensor systems would be required to exactly implement the control. However, most actuator and measurement hardware is discrete by nature, and only in rare cases can we write down the partial differential equations of motion for the structures, since most structures have very complicated geometry. Hence, in most cases, model reduction is inevitably introduced at the very beginning stage when the continuous system is modeled as a discrete system by using some discretization approach, for example, the finite element method. In addition to modeling, there is another necessity for order reduction due to the limitation of computer capacity and economy. Usually the finite element model of a real structure has thousands of degrees of freedom. Such a model is neither computationally economical nor "Riccati-solvable" for the controller design. Therefore, a further model reduction of the mathematical model is indispensable. But, when a controller that is designed based on the reduced model is applied to control the real structure, spillover of one form or another results.

A. Characteristics of Spillover

Let a structural dynamics system be partitioned into two subsystems, a controlled subplant and a residual subplant, in the form

$$\begin{bmatrix} M_c & M_{cr} \\ M_{rc} & M_r \end{bmatrix}\begin{Bmatrix} \ddot{x}_c \\ \ddot{x}_r \end{Bmatrix} + \begin{bmatrix} D_c & D_{cr} \\ D_{rc} & D_r \end{bmatrix}\begin{Bmatrix} \dot{x}_c \\ \dot{x}_r \end{Bmatrix} + \begin{bmatrix} K_c & K_{cr} \\ K_{rc} & K_r \end{bmatrix}\begin{Bmatrix} x_c \\ x_r \end{Bmatrix} = \begin{Bmatrix} P_c \\ P_r \end{Bmatrix} u$$
$$y = [V_c \quad V_r]\begin{Bmatrix} x_c \\ x_r \end{Bmatrix} + [W_c \quad W_r]\begin{Bmatrix} \dot{x}_c \\ \dot{x}_r \end{Bmatrix} \tag{16}$$

with subscripts c and r denoting controlled and residual subplant, respectively. Here the x_c and x_r vectors can be in physical coordinates or some generalized coordinates. The controlled subplant, for which the controller is designed, is

$$M_c \ddot{x}_c + D_c \dot{x}_c + K_c x_c = P_c u$$
$$y_c = V_c x_c + W_c \dot{x}_c \tag{17}$$

A general dynamic output feedback controller for the controlled subplant has the form

$$\dot{q} = A q + B y_c$$
$$u = G q \tag{18}$$

in which the controller system matrices A, B, and G are the design parameters. Note that the controller design is based on y_c, the output of the controlled subplant, although the actual measurement available is y. The controller is designed to stabilize the controlled subplant by using some existing control approach, for example LQ optimal control theory. If the size of the q vector is twice the x_c vector, it is the case of full-order state feedback.

Now, if the controller is applied to the full-order system, Eq. (16), the augmented closed-loop system can be described by the following first-order form:

$$\begin{bmatrix} I & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & I & \vdots & 0 & 0 \\ 0 & M_c & D_c & \vdots & M_{cr} & D_{cr} \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & I \\ 0 & M_{rc} & D_{rc} & \vdots & M_r & D_r \end{bmatrix}\begin{Bmatrix} \dot{q} \\ \ddot{x}_c \\ \dot{x}_c \\ \cdots \\ \ddot{x}_r \\ \dot{x}_r \end{Bmatrix} = \begin{bmatrix} A & BW_c & BV_c & \vdots & BW_r & BV_r \\ 0 & I & 0 & \vdots & 0 & 0 \\ P_c G & 0 & -K_c & \vdots & 0 & -K_{cr} \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & I & 0 \\ P_r G & 0 & -K_{rc} & \vdots & 0 & -K_r \end{bmatrix}\begin{Bmatrix} q \\ \dot{x}_c \\ x_c \\ \cdots \\ \dot{x}_r \\ x_r \end{Bmatrix} \tag{19}$$

From the preceding expression, we see that although the controller is intentionally designed to control the c subplant and leave the r subplant undisturbed, there are three channels through which the control energy is spilled over into the residual subplant. These three spillover factors are depicted in Fig. 1 and can be clearly identified in Eq. (16). Balas termed the P_r submatrix the control spillover factor, through which the control input u goes directly into the r subplant; V_r and W_r submatrices are the observation spillover terms, through which the measurement y is contaminated by the output from the r subplant. The third factor, recently referred to by Yam et al.[16] as the dynamic spillover, is due to the coupling submatrices M_{cr}, M_{rc}, D_{cr}, D_{rc}, K_{cr}, and K_{rc}. All three types of spillover have an effect on the closed-loop performance and stability.

Suppose there exists a coordinate transformation such that the aforementioned three spillover factors are eliminated. Then the closed-loop poles would be simply the union of the two diagonal submatrices in Eq. (19), i.e., the union of the poles of the closed-loop c subplant and the open-loop poles of the undisturbed r subplant. But this favorable situation occurs only when the full-order system is not a minimum realization or, in other words, it is not fully controllable and/or not fully observable. For instance, if the actuator and sensor distribution is in a configuration of the shape of some normal modes, then those corresponding modes can be controlled independently and leave the other modes undisturbed. However, this actuator/sensor configuration requires an infinite number of actuators and sensors and is not practical. In the usual case, there generally exists some spillover. The closed-loop poles, then, are perturbed from the designed locations by the nonzeros in the off-diagonal submatrices in Eq. (19). For instance, if the system is described in normal coordinates, which are commonly used, then the augmented closed-loop system equation, with the assumption that damping is proportional and $D_{cr} = D_{rc} = 0$, is in the form

$$\begin{bmatrix} I & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & I & \vdots & 0 & 0 \\ 0 & I & C_c & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & I \\ 0 & 0 & 0 & \vdots & I & 0 \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \ddot{x}_c \\ \dot{x}_c \\ \cdots \\ \ddot{x}_r \\ \dot{x}_r \end{Bmatrix} = \begin{bmatrix} A & BW_c & BV_c & \vdots & BW_r & BV_r \\ 0 & I & 0 & \vdots & 0 & 0 \\ P_cG & 0 & -\Omega_c^2 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & I & 0 \\ P_rG & 0 & 0 & \vdots & 0 & -\Omega_r^2 \end{bmatrix} \begin{Bmatrix} q \\ \dot{x}_c \\ x_c \\ \cdots \\ \dot{x}_r \\ x_r \end{Bmatrix}$$

in which P_rG, BV_r, and BW_r, the control and observation spillovers, are responsible for the pole perturbation. Although either control spillover or observation spillover alone has no effect on closed-loop stability, the existence of either of them can degrade the controller performance. Balas showed in Ref. 15 that the combined effect of control and observation spillover usually degrades the performance of the controller and sometimes can destabilize the closed-loop system. This can be effectively explained by the fact that large perturbation may move some of the poles to the right half plane.

Recently, Yam et al.[16] proposed a flexible system model reduction approach based on actuator and sensor influence functions. The transformed system equation is free from control and observation spillovers but has dynamic coupling. The advantage is that, unlike the normal mode formulation, the design parameter B and G matrices do not enter the perturbation, though the closed-loop system is still coupled by the dynamic spillovers M_{cr}, D_{cr}, and K_{cr}. In this case, a perturbation technique can be used effectively to study and estimate the closed-loop stability without the design parameters complicating the analysis. However, one feature of Yam's formulation that is not appealing is that the transformation has no physical significance and seems somewhat artificial. Besides, there is a shortage of certainty about the smallness of the dynamic spillover. Here, we propose a Krylov formulation for the control of flexible structures, which has some characteristics similar to Yam's formulation but which also has strong physical motivation.

B. Krylov Formulation for Control of Flexible Structures

As mentioned in Sec. II, the Krylov reduced model not only preserves the low-frequency moments but also presents the following attractive form

$$\begin{bmatrix} M_c & M_{cr} \\ M_{rc} & M_r \end{bmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{x}_r \end{Bmatrix} + \begin{bmatrix} D_c & D_{cr} \\ D_{rc} & D_r \end{bmatrix} \begin{Bmatrix} \dot{x}_c \\ \dot{x}_r \end{Bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} x_c \\ x_r \end{Bmatrix} = \begin{Bmatrix} P_c \\ 0 \end{Bmatrix} u$$

$$y = [V_c \quad 0] \begin{Bmatrix} x_c \\ x_r \end{Bmatrix} + [W_c \quad 0] \begin{Bmatrix} \dot{x}_c \\ \dot{x}_r \end{Bmatrix} \tag{20}$$

where

$$P_c = [P_1^T \ 0 \ \cdots \ 0]^T, \qquad V_c = [V_1 \ 0 \ \cdots \ 0]$$

$$W_c = [W_1 \ 0 \ \cdots \ 0]$$

There is no control or observation spillover in the transformed system equation. For general symmetric damping, the transformed mass and damping matrices are, in general, full; however, experience so far indicates that the coupling is usually small. Reference 17 has an example to show that the dynamic coupling in the Krylov formulation is small.

When Eq. (20) is combined with the feedback controller, Eq. (18), the augmented system can be expressed by

$$\begin{bmatrix} I & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & I & \vdots & 0 & 0 \\ 0 & M_c & D_c & \vdots & M_{cr} & D_{cr} \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & I \\ 0 & M_{rc} & D_{rc} & \vdots & M_r & D_r \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \ddot{x}_c \\ \dot{x}_c \\ \cdots \\ \ddot{x}_r \\ \dot{x}_r \end{Bmatrix} = \begin{bmatrix} A & BW_c & BV_c & \vdots & 0 & 0 \\ 0 & I & 0 & \vdots & 0 & 0 \\ P_cG & 0 & -I & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & I & 0 \\ 0 & 0 & 0 & \vdots & 0 & -I \end{bmatrix} \begin{Bmatrix} q \\ \dot{x}_c \\ x_c \\ \cdots \\ \dot{x}_r \\ x_r \end{Bmatrix} \tag{21}$$

which, like Yam's formulation, is coupled only by dynamic spillover.

IV. Examples

A. Model Reduction Example

The example considered is a plane truss structure similar to the one in Ref. 3 but on a smaller scale (see Fig. 2). The

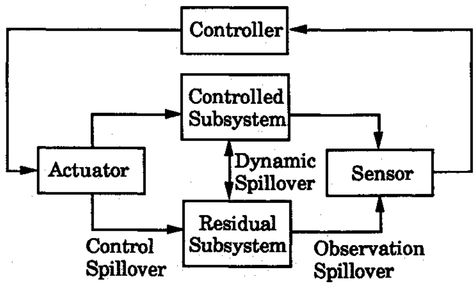


Fig. 1 Characteristics of spillover.

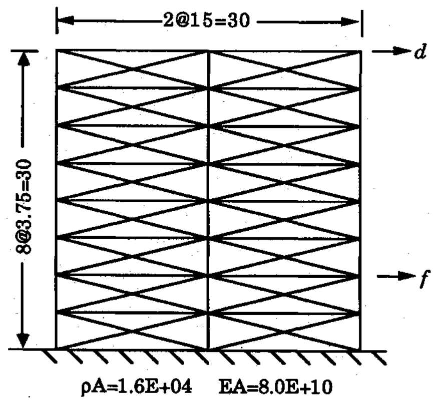


Fig. 2 Details of plane truss structure for model reduction example.

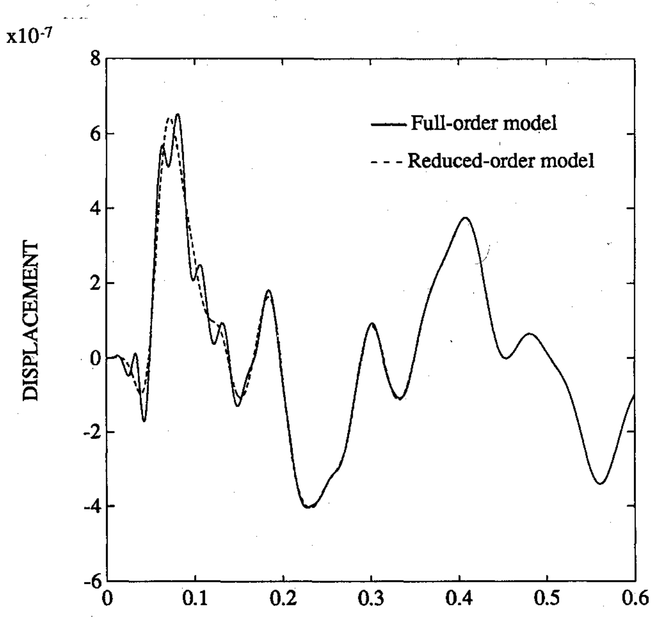


Fig. 3 Impulse response: eight normal modes and exact solution.

structure has 48 degrees of freedom and has a force actuator at f and a displacement sensor at d. The structure geometry is designed to provide closely spaced eigenvalues. A formula in Ref. 18 is used to generate a generalized proportional damping matrix such that modes 1–5 have a 3% damping ratio, modes 6–10 have a 5% damping ratio, and the remaining higher modes have successively higher damping. The structure is reduced to eight degrees of freedom by using eight Krylov vectors based on the algorithm in Sec. II. In this case, the reduced-order model matches the low-frequency moments $\hat{V}(\hat{K}^{-1}\hat{M})^i\hat{K}^{-1}\hat{P}$, for $i = 1, 2, \ldots, 7$, of the full-order structure. The other reduced-order model, obtained by using eight undamped Krylov vectors based on an algorithm for undamped systems,[6] is also examined. Figures 3–5 compare the accuracy of the impulse response of the normal-mode reduced

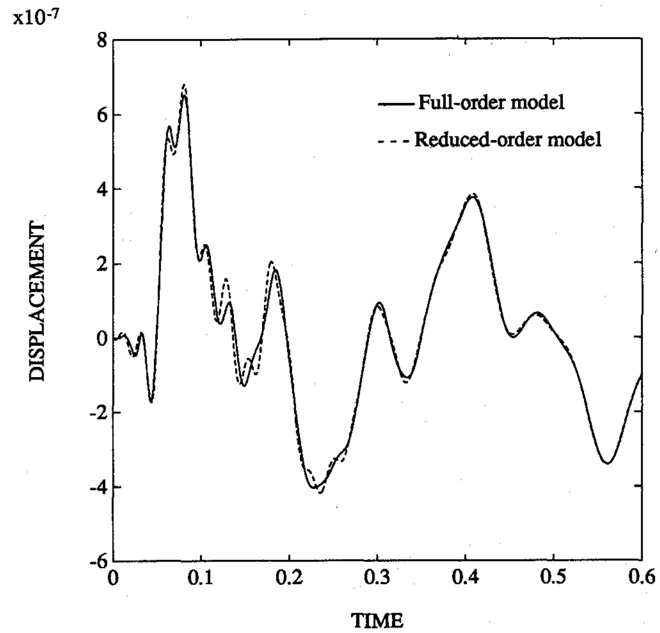


Fig. 4 Impulse response: eight damped Krylov modes and exact solution.

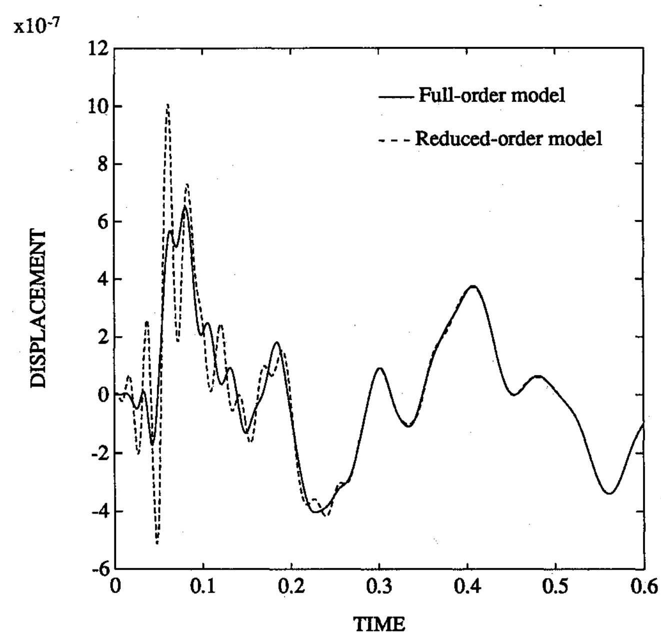


Fig. 5 Impulse response: eight undamped Krylov modes and exact solution.

model, the damped Krylov reduced model, and the undamped Krylov reduced model. It is seen that for this example the normal-mode reduced model and the damped Krylov reduced model have about the same accuracy, while the undamped Krylov reduced model is poor. The eigenvalues of the full-order system and the reduced-order models are compared in Table 1.

B. Flexible Structure Control Example

The example considered is a 20 degree-of-freedom lightly damped plane truss structure as shown in Fig. 6. The damping matrix is proportional such that modes 1–5 have a 0.1% damping ratio, modes 6–10 have a 0.2% damping ratio, and the remaining higher modes have successively higher damping up to about 0.5%. A force actuator is located at f and a displacement sensor is located at d. The actuator is contaminated by a zero-mean white noise with intensity 10^{-3}. The sensor is contaminated by a zero-mean white noise with intensity 10^{-12}.

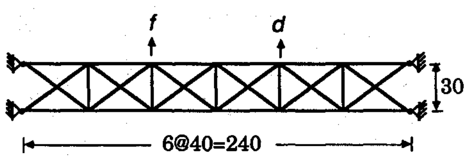
Table 1 Eigenvalues of full-order system and reduced-order models of model reduction example

Full-order system		Damped Krylov		Undamped Krylov	
real	imaginary	real	imaginary	real	imaginary
−5.80579E−1	1.93439E+1	−5.80579E−1	1.93439E+1	−5.80579E−1	1.93439E+1
−1.76955E+0	5.89584E+1	−1.76955E+0	5.89587E+1	−1.76955E+0	5.89583E+1
−3.05172E+0	1.01678E+2	−3.29455E+0	1.09487E+2	−3.24649E+0	1.05983E+2
−3.27286E+0	1.09046E+2	−4.56416E+0	1.34403E+2	−3.27363E+0	1.09065E+2
−4.22447E+0	1.40752E+2	−8.02594E+0	1.66038E+2	−9.30531E+0	1.85965E+2
−8.18679E+0	1.63531E+2	−9.10266E+0	1.92236E+2	−1.21170E+1	2.30196E+2
−9.29019E+0	1.85571E+2	−1.51837E+1	2.59751E+2	−1.90894E+1	2.91242E+2
−1.02178E+1	2.04099E+2	−1.63597E+1	2.69795E+2	−2.40600E+1	3.26846E+2
−1.02499E+1	2.04742E+2				
−1.12160E+1	2.24039E+2				
−1.23965E+1	2.35503E+2				
−1.33062E+1	2.43967E+2				
−1.36286E+1	2.46896E+2				
−1.36956E+1	2.47500E+2				
−1.38840E+1	2.49191E+2				
−1.40794E+1	2.50933E+2				

Table 2 Stability of the controllers

Controller	$\rho=0.05$	0.1	0.5	1.0	5.0	10.0	50.0	100.0	500.0
K2	U[a]	U	U	U	S	S	S	S	S
K4	U	S	S	S	S	S	S	S	S
K6	S[b]	S	S	S	S	S	S	S	S
K8	U	S	S	S	S	S	S	S	S
K10	S	S	S	S	S	S	S	S	S
N2	U	U	U	U	U	U	S	S	S
N4	U	U	U	U	U	U	S	S	S
N6	U	U	U	U	S	S	S	S	S
N8	S	S	S	S	S	S	S	S	S
N10	S	S	S	S	S	S	S	S	S

[a]U = unstable closed-loop system. [b]S = stable closed-loop system.



ρA=1 EA=1.0E+5

Fig. 6 Details of plane truss structure for control example.

The structure is reduced to low-order models by using either damped Krylov modes or normal modes: five Krylov reduced models with order 2, 4, 6, 8, and 10, respectively, and five normal-mode reduced models with order 2, 4, 6, 8, and 10, respectively. Based on each reduced-order model, a Linear Quadratic Gaussian (LQG) control design is carried out to minimize the performance index

$$J=\frac{1}{2}\lim_{t\to\infty} E\left(\dot{x}_c^T M_c \dot{x}_c + x_c^T K_c x_c + \rho u^T u\right)$$

in which the first two terms represent the total energy of the reduced system, and the third term represents the control cost. A positive scalar ρ is used to adjust the relative weighting of the regulation cost and control cost penalties. Overall controller authority, actuator mean-square force levels and controller bandwidth are all inversely proportional to ρ. The value of ρ was varied from 0.05 to 500 to study the closed-loop stability and controller performance. All the calculations and designs were performed by using the CTRL-C package run on a VAX computer. The results are summarized by Table 2 and Fig. 7, in which K2 stands for the controller designed based on the

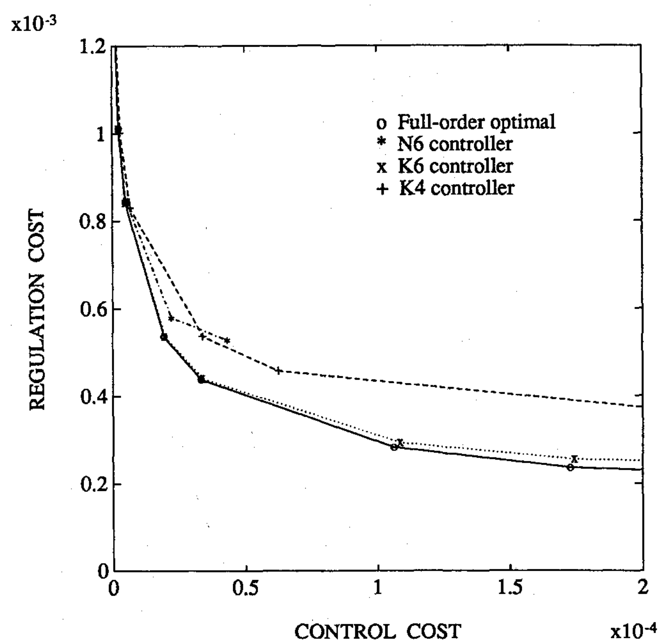


Fig. 7 Performance plot.

second-order Krylov reduced model, N2 stands for the controller designed based on the second-order normal-mode reduced model, and so on. From Table 2, it is seen that controllers designed using normal-mode reduced models are more likely to cause closed-loop instability than controllers designed using Krylov reduced models, especially when the controller bandwidth is large (small ρ) and the order of the reduced model is low. The performance of the controllers is presented in Fig. 7, which shows the regulation cost

$$J_e=\lim_{t\to\infty} E\left[\dot{x}^T M \dot{x} + x^T K x\right]$$

as a function of the control cost

$$J_e=\lim_{t\to\infty} E\left[u^T u\right]$$

(obtained by varying ρ). It is seen that the K6 controller has a performance very close to that of the full-order optimal LQG controller. The N6 controller, which fails to yield stable design for $\rho<5.0$, has a performance not very much better than that of the K4 controller. Performance curves of K8, K10, N8, and N10 are all very close to the optimal one and, hence, are not shown in Fig. 7.

V. Conclusions

Krylov vectors and the concept of parameter matching are combined together to develop model reduction algorithms for structural dynamics systems. The Krylov reduced models obtained are shown to have a promising aspect in the application to control of flexible structures. The formulation based on Krylov vectors can eliminate control and observation spillovers while leaving only the dynamic spillover terms to be considered. Two examples, one model order reduction example and one flexible structure control example, are provided to show the efficacy of the Krylov method.

Acknowledgments

This work was conducted under contract NAS9-17254 with the NASA Lyndon B. Johnson Space Center. The interest of Rodney Rocha is gratefully acknowledged.

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